

State Dependent Utility*

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Abstract

We propose a new approach to utilities that is consistent with state-dependent utilities. In our model utilities reflect the level of consumption satisfaction of flows of cash in future times as they are valued when the economic agents are making their consumption and investment decisions. The theoretical framework used for the model is one proposed by the author in *Dynamic State Tameless* ([arXiv:math.PR/0509139](https://arxiv.org/abs/math.PR/0509139)). The proposed framework is a generalization of the theory of Brownian flows and can be applied to those processes that are the solutions of classical Itô stochastic differential equations, even when the volatilities and drifts are just locally δ -Hölder continuous for some $\delta > 0$. We develop the martingale methodology for the solution of the problem of optimal consumption and investment. Complete solutions of the optimal consumption and portfolio problem are obtained in a very general setting which includes several functional forms for utilities in the current literature, and consider general restrictions on minimal wealths. As a secondary result we obtain a suitable representation for straightforward numerical computations of the optimal consumption and investment strategies.

1 Introduction

The problem of optimal consumption and investment for a “small investor” whose actions do not influence market prices is at the core of portfolio management and it is the building block for the development of equilibrium theory. The modern treatment of this problem when asset prices follows Itô processes started with the seminal works of Merton [38] and Merton [39]. Using a “martingale” approach, Cox and Huang [10], and Karatzas et al. [23] solved the problem in

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more general settings in the case of complete markets. A representation formula is derived in Ocone and Karatzas [40] in terms of expectation of random variables which involve Malliavin derivatives of the coefficients of the model. The latter gives theoretical formulas for optimal portfolios and consumption strategies.

In order to obtain numerical representations for the structure of the optimal portfolios and consumption processes it is natural to use methods based on the dynamic-programming approach. However numerical schemes based on PDEs become increasingly difficult to evaluate when the dimension of the underlying state variable increase and even standard techniques are somehow inappropriate for the solution of the PDE that arises in small dimensions (Dangl and Wirl [13]). As a result attention has been directed to models admitting closed form solutions (i.e. Watchter [42], Kim and Omberg [27], Lioui and Poncet [31]), specifications which are computationally tractable based on dynamic programming techniques (Brennan et al. [3], Brennan [2], Brennan and Xia [4], Campbell et al. [5]), discrete time models based on approximated Euler equations (Balduzzi and Lynch [1], Dammon et al. [12], Campbell and Viceira [6], and Campbell and Viceira [7]) or Monte Carlo techniques (Cvitanic et al. [11] and Detemple et al. [16]).

However, the main drawback of the standard models for optimal consumption and investment is their lack of agreement with empirical data. These inconsistencies are documented with the name of several puzzles such as the “equity premium puzzle” (Mehra and Prescott [36]), the “risk-free rate puzzle” (Weil [43]), and “risk-aversion puzzle” (Jackwerth [22]). In order to address these problems several generalizations have been proposed. One of these models habit formation for consumers. Some examples are Constantinides [9], Hindy and Huang [19], Hindy et al. [20], Hindya et al. [21]. Another approach is the construction of recursive utility. Some references for this are Duffie and Epstein [17], Epstein and Zin [18] and Lazrak and Quenez [30]. A different way to account for the discrepancy of theory and empirical data is the assumption of transaction costs for changes in consumption levels. Some references for this approach are Magill and Constantinides [35], Shreve and Soner [41], Davis and Norman [15] to cite a few.

In fact one of the reasons why the standard utility models fail to fit economic behavior might be the fact that state independent utilities are not appropriate for modeling the behavior of human beings. For instance, see Karni [26], Karni [25]. Partly motivated by the above, some literature in finance has focused on state-dependent utilities to explain the behavior of individual consumers and investors and of financial variables. Some recent references are Chabi-Yo et al. [8], Melino and Yang [37] and Dantine et al. [14] among others.

In this paper we propose a new approach for utilities. Mathematically it looks similar to the standard model for utilities, but we interpreted it in a way that is consistent with state-dependent utilities. The traditional approach is to consider that utilities reflect the level of “happiness” for consumption levels in the future (discounted by the value of money in a bank account). See Karatzas and Shreve [24]. We believe that in complete markets where an agent can hedge

any flow of money, this view is inappropriate. To simplify things let us assume that a time horizon $[0, T]$ is fixed, and 0 is the time when the agent is making his decision on an optimal consumption level and investment and there is not any preference for terminal wealth. *Our guiding principle is the believe that agents have utilities for consumption of flows of money in future times as they are valued (by the market) at the time when they are making their consumption and investment decisions.* Another way to look at this, is that people tend to value things according to their social and economic context, instead of just looking at quantitative values. For instance people tend to appreciate more the ability to have enough money to pay off their debts in depression times than the ability to buy luxuries in good times. The above remark changes completely the optimization problem to consider, with the advantage that most of the tools used to solve the old problem can be used in this setting. In particular the martingale methodology is available. A consequence of adopting this approach is that complete solutions of the optimal consumption and portfolio problem are obtained in a very general setting that includes several of the functional forms for utilities in the literature, and considers quite general restrictions of minimal wealth. As a secondary result we obtain suitable representation for straight-forward numerical computations of the optimal consumption and *investment* strategies.

The theoretical framework used to solve the above problem is the one proposed in Londoño [34]. In this introduction we just named the processes described in the cited paper as consistent measurable processes; these are processes whose evolution between any two times only depends on *the evolution* of the underlying Brownian motion and satisfies some consistency conditions. The methodology described can be used in processes that are a generalization of Brownian flows (Kunita [28]), and is applicable to those processes that are the solutions of classical Itô stochastic differential equations, even when the volatilities and drifts are just locally δ -Hölder continuous for some $\delta > 0$. It can also be straightforwardly adapted to stochastic volatility models whose underlying drifts and volatilities evolution are described by classical stochastic Itô differential equations. As discussed in Londoño [34] the theoretical framework is potentially useful when the underlying randomness is generated by a (not necessarily continuous) Lévy process. At the same time the theoretical framework described in Londoño [34] is a particular case of the theory of arbitrage and valuation presented in Londoño [32]. To the best of our knowledge this theory of arbitrage and valuation is the most general existing theory in the case of (continuous) semimartingales driven by Brownian filtrations with continuous coefficients.

A brief explanation of the heuristics of using consistent measurable processes in finance (in the problem of optimal consumption and investment) is given below. When an agent enters an economy, it would be desirable that any decision made during his lifetime would depend only on events that are happening within the time framework defined by the current time and the entering time, and on the state of the economy when he enters; we hope to be able to show to the reader that this is the case throughout this paper. Although this might be a

simplification that is likely not to be truthful, in the long run events in the past (before the agent entered the economy) become neglectful. Even more important, past aware models necessarily make the entering time a privileged one; the entering time would be the only time when no past information is required (besides the information contained in the current state of the model). Moreover, it would imply that the decision making process of agents would depend on their entering time, regardless of identical preferences over their lifetime. The previous remark also applies to the standard basic time-additive utility function maximization model (Karatzas and Shreve [24]). Let us expand on this point further. Assume that two agents decide to start investing in the market at two different times, let's say the first agent starts investing at time $t = 0$, and the second agent starts investing at time $t = 1$; assume that both agents have the same live expectation (to simplify things, assume that both are planing to leave the economy at time $t = 2$), and they do not have any additional source of income. Moreover, assume that both agents have identical preferences over consumption partners for times $1 \leq t \leq 2$, and at time $t = 1$ they have identical amounts of wealth. We believe that any reasonable model should produce identical levels of consumption and investment, assuming that they are the result of some kind of optimization procedure. Unfortunately, this is not the case for the basic time-additive utility function maximization model (Karatzas and Shreve [24]) when the processes are allowed to be general semimartingales. In fact, when an agent chooses his optimal consumption process at time 0 over the time framework $[0, 2]$, the agent is free to use portfolios and consumption processes that at any time “know” all the previous information between the given time and the initial time ($t = 0$). In particular the consumption and portfolio processes for times $t \in [1, 2]$ would be “aware” of all the information before time $t = 1$, and therefore *a priori* they would be different from the ones that are the result of the optimization procedure that the second agent is making at time $t = 1$ (with no previous information gathered). Note that the above does not hold for models which are solutions of stochastic differential equations whose coefficients are deterministic functions.

Next we describe the contents of the paper. In section 2 we review the model and definitions presented in Londoño [34], and review the definitions of utility that we use in this paper. In section 3 we present a martingale methodology needed to address the cited problem for the model described in Londoño [34], that plays the role of the martingale methodology of Cox and Huang [10], Karatzas et al. [23] and Ocone and Karatzas [40] in the current context. Finally, in section 4 we present the main results on optimal consumption and investment.

2 The model

First we introduce some notation which will be frequently used in this paper. Let $\mathbb{D} \subset \mathbb{R}^k$ be a open connected set. Let m be a non-negative integer. We denote by $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ the the Fréchet space of m -times continuous differentiable functions whose m -order derivatives are δ -Hölder continuous with semi-norms

$\|f\|_{m,\delta;K}$ defined in Kunita [29, Section 3.1] where $K \subset \mathbb{D}$ is a compact set and $0 \leq \delta \leq 1$. In case $m = 0$ (or $\delta = 0$) we denote $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ simply by $C^\delta(\mathbb{D}; \mathbb{R}^n)$ ($C^m(\mathbb{D}; \mathbb{R}^n)$).

We assume a d -dimensional Brownian Motion $\{W(t), \mathcal{F}_t; 0 \leq t \leq T\}$ starting at 0 and defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ where $\mathcal{F} = \mathcal{F}_T$ and $\{\mathcal{F}_t, 0 \leq t \leq T\}$ is the \mathbf{P} augmentation by the null sets of the natural filtration $\mathcal{F}_t^W = \sigma(W(s), 0 \leq s \leq t)$. Let $(\mathcal{F}_{s,t}) = \{\mathcal{F}_{s,t}, 0 \leq s \leq t \leq T\}$ be the two parameter filtration where $\mathcal{F}_{s,t}$ is the smallest sub σ -field containing all null sets and $\sigma(W_s(u) \mid s \leq u \leq t)$, where $W_s(u) \equiv W(u) - W(s)$. For each $0 \leq s \leq T$ we also define the σ -field \mathcal{P}_s of progressive measurable sets after time s as the σ -field of sets $P \in \mathcal{B}([s, T]) \otimes \mathcal{F}_{s,T}$, the product σ -field, such that $\chi_P(t, \omega)$, $t \geq s$, is a $\mathcal{F}_{s,t}$ progressive measurable (in t) process, where χ is the indicator function.

We denote by μ_s the measure on \mathcal{P}_s defined by $\mu_s(P) = \mathbf{E} \int_s^T \chi_P(s, \omega) dt$.

For definitions of consistent processes see Londoño [32]. However for the sake of completeness we review these definitions here. Let $\varphi(s, t, x, \omega)$, $0 \leq s \leq t \leq T$, $x \in \mathbb{D}$ be a \mathbb{R}^n -valued random field on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We call it a *continuous* $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ -*semimartingale* if $\varphi_s: t \rightarrow \varphi(s, t, \cdot)$, is a measurable random field with values in $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$, that is a continuous $(\mathcal{F}_{s,\cdot})$ semimartingale process decomposed as $\varphi(s, t, x) = \varphi_{loc}(s, t, x) + \varphi_{fv}(s, t, x)$, where $\varphi_{loc}(s, \cdot, \cdot)$ is a continuous $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ -local-martingale, and $\varphi_{fv}(s, \cdot, \cdot)$ is a continuous $C^{m,\delta}(\mathbb{D}; \mathbb{R}^n)$ -process of bounded variation for each $0 \leq s \leq T$. A pair (a, b) where $a(s, t, x, y)$ and $b(s, t, x)$ are measurable random fields $\mathcal{F}_{s,t}$ -progressive measurable in t , for all $x, y \in \mathbb{D}$, $0 \leq s \leq T$, is said to be the *local characteristics* of φ , if $(a(s, \cdot, x, y), b(s, \cdot, x))$ is the local characteristic of $\varphi_s \equiv \varphi(s, \cdot, \cdot)$ (see Kunita [29]) for any $s \leq t \leq T$. In addition, a pair (σ, b) where $\sigma(s, t, x)$ is a measurable random field with values in $L(\mathbb{R}^d; \mathbb{R}^n)$, where $L(\mathbb{R}^d; \mathbb{R}^n)$ denotes the set of matrices with size $n \times d$, $(\mathcal{F}_{s,t})$ -progressive measurable in t , for all $x \in \mathbb{D}$, $0 \leq s \leq T$, and b is as above is said to be the *volatility and drift processes* of φ if

$$\varphi_{loc}(s, t, x)(\omega) = \int_s^t \sigma(s, u, x) dW_s(u),$$

for all x, s, t and ω . If $b(s, \cdot, \cdot)$ and $\sigma(s, \cdot, \cdot)$ are processes of class $C^{m,\delta}$ for all $0 \leq s \leq T$, we shall say that φ has *volatility and drift of class* $C^{m,\delta}$.

Let $\varphi(s, t, x)$ and $\psi(s, t, x)$ be continuous $C(\mathbb{D}; \mathbb{R}^n)$ and $C(\mathbb{D}; \mathbb{D})$ semimartingales, respectively; in addition, it is assumed that $\psi(s, s, x) = x$ for all $x \in \mathbb{D}$, and $0 \leq s \leq T$. We say that the process φ is a *ψ -consistent semimartingale process* if for each $0 \leq s \leq s' \leq T$ there exists a set $N_{s,s'} \in \mathcal{P}_{s'}$ with $\mu_{s'}(N_{s,s'}) = 0$, such that $\varphi(s, t, x) = \varphi(s', t, \psi(s, s', x))$ for all $(t, \omega) \notin N_{s,s'}$ and all $x \in \mathbb{D}$. We say that the process φ is a *consistent semimartingale process* if φ is a φ -consistent process.

We assume $n + 1$ stocks whose *evolution price process* P is a consistent $C(\mathbb{R}_+^{n+1}; \mathbb{R}_+^{n+1})$ -semimartingale. For $0 \leq i \leq n$ we define the *price per-share process for the i -stock*, P_i , to be the P -consistent $C(\mathbb{R}_+^{n+1}; \mathbb{R}_+)$ -semimartingale process $P_i = \{P_i(s, t, p) = \pi_i \circ P(s, t, p), p \in \mathbb{R}^{n+1}, 0 \leq s \leq t \leq T\}$ where π_i de-

notes the projection on the i -component. We assume consistent progressive measurable P -consistent processes $\sigma_{i,j}$, b_i , δ_i , r , and θ_i of class $C^\delta(\mathbb{R}_+^{n+1}; \mathbb{R})$ for some $\delta > 0$, where \mathbb{R}_+ denotes the set of real positive numbers. It is assumed that $\sigma_{i,j}$, b_i , δ_i , r , and θ_i relate to P through the following stochastic differential equations

$$dP_i(s, t, p) = P_i(s, t, p) \left[b_i(s, t, p) dt + \sum_{1 \leq j \leq d} \sigma_{ij}(s, t, p) dW_s^j(t) \right]$$

$$P_i(s, s, p) = p_i, i = 1, \dots, n$$

where $W_s^j(t) = W^j(t) - W^j(s)$, and,

$$dP_0(s, t, p) = P_0(s, t, p) \left[-r(s, t, p) dt - \sum_{1 \leq j \leq d} \theta_j(s, t, p) dW_s^j(t) \right]$$

$$P_0(s, s, p) = p_0.$$

Throughout this paper we shall assume that $\theta(s, \cdot, p) \in \ker^\perp(\sigma(s, \cdot, p))$, (where $\ker^\perp(\sigma(s, \cdot, p))$ denotes the orthogonal complement of the kernel of $\sigma(s, \cdot, p)$) and,

$$b(s, t, p) + \delta(s, t, p) - r(s, t, p)\mathbf{1}_n = \sigma(s, t, p)\theta(s, t, p)$$

a.e. μ_s , for all $p \in \mathbb{R}_+^{n+1}$, and $0 \leq s \leq T$, where $\mathbf{1}'_n = (1, \dots, 1) \in \mathbb{R}^n$. This latter assumption implies that there are not state-tame arbitrage opportunities (see Londoño [32]).

The process of bounded variation $B = \{B(s, t, p)\}$, whose evolution $B(s, \cdot, p)$, $p \in \mathbb{R}_+^{n+1}$, $0 \leq s \leq T$ is given by the stochastic differential equation

$$dB(s, t, p) = B(s, t, p)r(s, t, p)dt, \quad B(s, s, p) = 1, \text{ for } 0 \leq s \leq t \leq T$$

will be called the *bond price process*.

We shall say that $\mathcal{M} = (P, b, \sigma, \delta, r, p^0)$ is a *financial market with terminal time T and initial time 0*, if $b = (b_1, \dots, b_n)$ is a vector of rate of return processes, $\sigma = (\sigma_{i,j})$ is a matrix of volatility coefficient processes, $\delta = (\delta_1, \dots, \delta_n)$ is a vector of dividend rate processes, r is an interest rate process, and $\theta'(s, t, p) = (\theta_1(s, t), \dots, \theta_d(s, t))$ is the *market price of risk*, and $p^0 \in \mathbb{R}_+^{n+1}$ is a vector of initial prices.

We define the *state price density process* to be the continuous $C(\mathbb{R}_+^{n+1}; \mathbb{R}_+)$ -semimartingale process defined by

$$H(s, t, p) = B^{-1}(s, t, p)Z(s, t, p) \quad \text{for } p \in \mathbb{R}_+^{n+1}, 0 \leq s \leq t \leq T$$

where

$$Z(s, t, p) = \exp \left\{ - \int_s^t \theta'(s, u, p) dW_s(u) - \frac{1}{2} \int_s^t \|\theta(s, u, p)\|^2 du \right\}$$

for $0 \leq s \leq t \leq T$, and $B^{-1}(s, t, p) = 1/B(s, t, p)$.

Assume that $\tau = \{\tau_s(x, p); 0 \leq s \leq T, x \in \mathbb{R}, p \in \mathbb{R}_+^{n+1}\}$ is a measurable family of stopping times. A *wealth structure* is a triple (X, τ, x_0) , where $x_0 \in \mathbb{R}$, and $X = \{X(s, t, x, p); x \in \mathbb{R}, p \in \mathbb{R}_+^{n+1}, 0 \leq s \leq t \leq \tau_s(x, p)\}$, is a family of continuous semimartingale processes with the property that $((X, P), \tau)$ is a consistent stopping structure where (X, P) is defined as

$$(X, P) = \{(X(s, t, x, p), P(s, t, p)), x \in \mathbb{R}, p \in \mathbb{R}_+^{n+1}, s \leq t \leq \tau_s(x, p)\}.$$

It is also assumed that the drift and volatility of the process $((X, P), \tau)$ is of class C^δ for some $\delta = \delta^X > 0$. We say that x_0 is the *initial value for the wealth process*, and we say that (X, τ) is a *wealth evolution structure*; we shall denote this by writing $(X, \tau) \in \mathcal{X}(\mathcal{M})$. For the definition of consistent stopping structure, measurable family of stopping times, and related ones, see Londoño [34].

For the sake of completeness we also review the definition for consistent stopping structure. Let $\tau = \{\tau(s, x), x \in \mathbb{D}, 0 \leq s \leq T\}$ be a family of stopping times with values in $[0, T]$, and semimartingales processes with drift and volatility of class $C^{m, \delta}$. It is assumed that for each $0 \leq s \leq T$, $x \in \mathbb{D}$, $\tau(s, x)$ is a stopping time relative to the filtration $\{\mathcal{F}_{s, t}; s \leq t \leq T\}$ with values in $[s, T]$, and that $\tau(s, x)(\omega)$ is a measurable random field that is lower semi-continuous with respect to (s, x) . Assume that $\psi(s, t, x), 0 \leq s \leq t \leq \tau(s, x), x \in \mathbb{D}$ is a family of process with values in \mathbb{R}^k , with $\psi_\tau = \{\psi(s, \tau(s, x) \wedge t, x), 0 \leq t \leq T; x, 0 \leq s \leq T\}$ where $s \wedge t = \min\{s, t\}$ is a $C(\mathbb{D}; \mathbb{R}^k)$ -semimartingale. We shall say that (ψ, τ) is a *consistent stopping structure* if for each $0 \leq s \leq T$ there exist $N_s \in \mathcal{P}_s$, $\mu_s(N_s) = 0$ with $\tau_s(x) = \tau_{t \wedge \tau}(\psi(s, t \wedge \tau, x))$ for all $(t, \omega) \notin N_s$ and all x . We say that the consistent stopping structure (ψ, τ) is of class $C^{m, \delta}(\mathbb{D}; \mathbb{R}^k)$ if ψ_τ is a process of class $C^{m, \delta}(\mathbb{D}; \mathbb{R}^k)$. Given a consistent stopping structure (ψ, τ) , we say that a family of \mathbb{R}^n -valued processes $\varphi = \{\varphi(s, t, x), s \leq t \leq \tau_s(x); x \in \mathbb{D}, 0 \leq s \leq T\}$ is a *ψ -consistent process with random time τ* , if φ_τ is a ψ_τ -consistent measurable process with two parameters. Similarly, we say that φ is a process of class $C^{m', \delta'}(\mathbb{D}; \mathbb{R}^n)$ if φ_τ is a process of the same class.

Let Γ be a continuous semimartingale process with random time τ with drift and volatility of class C^δ (where the positive number δ depends on Γ) with the property that $\Gamma(s, s, x, p) = 0$ and

$$\Gamma(s, t', x, p) + \Gamma(t', t, X(s, t', x, p), P(s, t', p)) = \Gamma(s, t, x, p)$$

for all $x \in \mathbb{R}$, $p \in \mathbb{R}_+^{n+1}$, and $0 \leq s \leq \tau_s(x, p)$. We say that a process Γ as above is an *income evolution structure for the wealth evolution structure* (X, τ) , and we say that (X, Γ, τ) is a *wealth and income evolution structure*. If $\Gamma(s, t, x, p) \leq 0$ for all $x, p, s \leq t \leq \tau_s(x, p)$ we say that Γ is a *consumption evolution structure for the wealth evolution structure* (X, τ) . Let $(\pi_0, \pi) = \{(\pi_0(s, t, x, p), \pi(s, t, x, p)); x \in \mathbb{R}, p \in \mathbb{R}_+^{n+1}, 0 \leq s \leq t \leq \tau_s(x, p)\}$ be a (X, P) -consistent progressive measurable process of class C^δ for some $\delta > 0$

with random time τ , and $\pi_0 + \pi' \mathbf{1}_n = X$ satisfying

$$\begin{aligned} B^{-1}(s, t, p)X(s, t, x, p) &= x + \int_s^t B^{-1}(s, u, p) d\Gamma(s, u, x, p) \\ &\quad + \int_s^t B^{-1}(s, u, p)\pi'(s, u, x, p)\sigma(s, u, p) dW_s(u) \\ &\quad + \int_s^t B^{-1}(s, u, p)\pi'(s, u, x, p)(b(s, u, p) + \delta(s, u, p) - r(s, u, p)\mathbf{1}_n) du \end{aligned}$$

for all $x \in \mathbb{R}$, $0 \leq s \leq t \leq \tau_s(x, p)$, $p \in \mathbb{R}_+^{n+1}$. We say that $((\pi_0, \pi), \Gamma, \tau)$ as above is a *portfolio evolution structure with random time τ , financed by the income Γ* . We say that a *wealth evolution structure $(X, \tau) \in \mathcal{X}(\mathcal{M})$ is financed by the income structure Γ* , if there exists a portfolio evolution structure $((\pi_0, \pi), \Gamma, \tau)$ with random time τ with $\pi_0 + \pi' \mathbf{1}_n = X$. In this case we say that (X, Γ, τ) is a *hedgeable wealth-income structure*.

Next we discuss the concept of utility that we shall use in this paper.

Definition 1. Consider a function $U: (0, \infty) \mapsto \mathbb{R}$ continuous, strictly increasing, strictly concave and continuous differentiable, with $U'(\infty) = \lim_{x \rightarrow \infty} U'(x) = 0$ and $U'(0+) \triangleq \lim_{x \downarrow 0} U'(x) = \infty$. Such a function will be called a *utility function*.

Classic examples of utility functions are $U_\alpha(x) = x^\alpha/\alpha$ for some $\alpha \in (0, 1)$, $0 \leq x < \infty$, and $U(x) = \log(x)$. For every utility function $U(\cdot)$, we shall denote by $I(\cdot)$ the inverse of the derivative $U'(\cdot)$; both of these functions are continuous, strictly decreasing and map $(0, \infty)$ onto itself with $I(0+) = U'(0+) = \lim_{x \rightarrow 0+} U'(x) = \infty$, $I(\infty) = \lim_{x \rightarrow \infty} I(x) = U'(\infty) = 0$. We extend U by $U(0) = U(0+)$, and we keep the same notation to the extension to $[0, \infty)$ of U hopping that it would be clear to the reader to which function we are referring. It is a well known result that

$$\max_{0 < x < \infty} (U(x) - xy) = U(I(y)) - yI(y), \quad 0 < y < \infty \quad (1)$$

Definition 2. Consider a continuous function $U_1: [0, T] \times (0, \infty) \mapsto \mathbb{R}$, such that $U_1(t, \cdot)$ is a utility function in the sense of Definition 1 for all $t \in [0, T]$. It follows that $I_1(t, x) \triangleq (\partial U_1(t, x)/\partial x)^{-1}$, the inverse of the derivative of U_1 , is a continuous function. Similarly if a utility function $U_2: (0, \infty) \mapsto \mathbb{R}$ is given then $I_2(t, x) \triangleq (\partial U_2(t, x)/\partial x)^{-1}$ is continuous. Let us denote

$$\mathcal{X}(t, y) \triangleq I_2(y) + \int_t^T I_1(t', y) dt'. \quad (2)$$

We shall call a couple of functions as above a *state preference structure*.

Under the conditions outlined in the previous definition, it is easy to see that $\mathcal{X}: [0, T] \times (0, \infty) \rightarrow (0, \infty)$ is a continuous function with the property that

for each t , $\mathcal{X}(t, \cdot)$ maps $(0, \infty)$ onto itself, is strictly decreasing with $\mathcal{X}(t, 0+) = \lim_{y \downarrow 0} \mathcal{X}(t, y) = \infty$ and $\mathcal{X}(t, \infty) = \lim_{y \rightarrow \infty} \mathcal{X}(t, y) = 0$.

We extend U_1 and U_2 by defining $U_1(t, 0) = U(t, 0+)$, for all $0 \leq t \leq T$ and $U_2(0) = U_2(0+)$, and we keep the same notation to the extension of U_1 to $[0, T] \times [0, \infty)$, and the extension of U_2 to $[0, \infty)$. We hope that it would be clear to the reader to which function we are referring.

We point out that \mathcal{X}^{-1} defined for each t as $\mathcal{X}^{-1}(t, \cdot)$, the inverse of $\mathcal{X}(t, \cdot)$, share the same properties to the some mentioned for \mathcal{X} . We next discuss the meaning of those utility functions defined above. We should interpret $U_1(t, x)$, for $t \in [0, T]$ the level of “happiness” for an agent consuming x units of wealth per unit of time at time t , as valued at time 0, when the agent is planning its consumption. Similarly, we should understand for $U_2(x)$ the level of “happiness” for an agent having a final wealth of x units (at time T) as valued at time 0. This is contrary with the traditional approach where an agent has preferences on their consumption behavior according to their value as discounted by a bank account, and is closer in approach to a utility function that is state dependent. See the literature on state dependent utilities cited above.

For $s \leq t$ define $\alpha(s, t) = \mathcal{X}(s, \mathcal{X}^{-1}(t, \cdot))$. Then $\alpha(s, t) = \alpha(s, t') \circ \alpha(t', t)$ for all s, t , and t' in $[0, T]$, where \circ denotes standard composition of functions. We also observe that if $\alpha^I(s, t) \triangleq I_1(s, \mathcal{X}^{-1}(t, \cdot))$ then $\alpha^I(s, t) \circ \alpha(t, s) = \alpha^I(s, s)$. Throughout this paper we shall assume the following condition on the utility structure.

Condition 1 (Homogeneity). *Let (U_1, U_2) be a state preference structure defined as above. For all $s, t \in [0, T]$ there exist constants $\alpha_{s,t}$ and α_s^I such that $\alpha(s, t)(x) = \alpha_{s,t}x$, and $\alpha^I(s, s)(x) = \alpha_s^I x$ where $\alpha(s, t)$ and $\alpha^I(s, t)$ are defined as the previous paragraph. In this case we say that (U_1, U_2) is a homogeneous state preference structure.*

A way to see this is to say that the structure for the utility preferences remains the same as time evolves. We next describe some important examples that fit the previous conditions.

Example 1. Assume a continuous positive function $h: [0, T] \rightarrow (0, \infty)$, and assume that $U_1(t, x) = x^\alpha h(t)$ and $U_2(x) = cx^\alpha$ with $\alpha \in (0, 1)$ and $c \geq 0$. This is an state preference structure that satisfies Condition 1. Indeed in this case

$$\alpha_{s,t} = \frac{c^{1/(1-\alpha)} + \int_s^T h^{1/(1-\alpha)}(t') dt'}{c^{1/(1-\alpha)} + \int_t^T h^{1/(1-\alpha)}(t') dt'}, \quad \alpha_t^I = \frac{h^{1/(1-\alpha)}(t)}{c^{1/(1-\alpha)} + \int_t^T h^{1/(1-\alpha)}(t') dt'}$$

Example 2. Assume a continuous positive function h as above, and assume that $U_1(t, x) = h(t) \log(x)$ and $U_2(x) = c \log(x)$, with $c \geq 0$. It follows that this is a state preference structure that satisfies Condition 1 with

$$\alpha_{s,t} = \frac{c + \int_s^T h(t') dt'}{c + \int_t^T h(t') dt'}, \quad \alpha_t^I = \frac{h(t)}{c + \int_t^T h(t') dt'}$$

Example 3. Let $U_1(t, x) = h(t)u(x/h(t))$ and $U_2(x) = cu(x/c)$, where $u(\cdot)$ is a utility function, $h(\cdot)$ is a positive continuous function and $c > 0$. It follows that (U_1, U_2) is a state preference structure that satisfies Condition 1. In this case

$$\alpha_{s,t} = \frac{c + \int_s^T h(t') dt'}{c + \int_t^T h(t') dt'}, \quad \alpha_t^I = \frac{h(t)}{c + \int_t^T h(t') dt'}$$

In particular, when $h \equiv 1$ we obtain that $U_1(t, x) = u(x)$, and $U_2(x) = cu(c^{-1}x)$ for some $c > 0$ define a state preference structure that satisfies Condition 1.

3 A Martingale approach

Definition 3. Assume that $(X, \Gamma, \tau) \in \mathcal{X}(\mathcal{M})$ is a wealth-income evolution structure. Assume that $\Gamma \equiv E - C$, where

$$dC(s, t, x, p) = c(s, t, x, p) dt \quad (3)$$

and,

$$dE(s, t, p) = \varepsilon(s, t, p) dt \quad (4)$$

for non-negative (X, P) consistent processes c and ε of class C^δ for some $\delta > 0$. Moreover assume that

$$\mathbf{E} \left[\int_s^T H(s, u, p) \varepsilon(s, u, p) du \right] < \infty$$

for all $p \in \mathbb{R}_+^n$, and $0 \leq s \leq T$. Similarly assume that

$$\mathbf{E} \left[\int_s^T H(s, u, p) c(s, u, x, p) du \right] < \infty$$

for all $p \in \mathbb{R}_+^n$, $x \in \mathbb{R}$ and $0 \leq s \leq T$.

We should say $(X, c, \varepsilon, \tau)$ as above is a rate of consumption and endowment evolution structure. We shall say that c is the consumption rate evolution structure, and ε is the endowment rate evolution structure. We also say that E is a cumulative endowment structure, and C is a cumulative consumption structure.

A minimal wealth structure L is a P consistent process with drift and volatility of class C^δ for some $\delta > 0$ where $L(s, \cdot, p)H(s, \cdot, p)$ is uniformly bounded below for all p, s , (where the bound might depend on p and s) such that

$$\mathbf{E} [H(s, t, p)L(s, t, p)] < \infty$$

for all p, s and t .

We should emphasize that the name might be confusing, since a minimal wealth structure is *not* a wealth structure in the sense defined above. However

we keep the name for consistency with standard use. It is natural to believe that the evolution of income due to labor only depends on the evolution of the state of the economy and not on the current wealth of an agent.

Typically we are interested in consumption and endowment structure evolution structures whose wealth remains above some given process. Next we present the definition that embodies this idea.

Definition 4. Let (X, ε, c, T) (denoted by (X, ε, c)) be a hedgeable (by a state tame portfolio) cumulative consumption and endowment evolution structure, as in Definition 3, with portfolio evolution structure (π_0, π) . We shall say that the couple (π, c) of portfolio on stocks and rate of consumption, is admissible for (L, ε) and write $(\pi, c) \in \mathcal{A}(L, \varepsilon)$ if for any x, s and p with $x \geq L(s, s, p)$

$$X(s, t, x, p) \geq L(s, t, p) \quad a.e. \quad (5)$$

If there is not portfolio on stocks and rate of consumption for (L, ε) we should say that the class cited above is empty, and we would denote this by $\mathcal{A}(L, \varepsilon) = \emptyset$

For any hedgeable wealth and income evolution structure $(X, E - C)$ with (π, c) admissible for (L, ε) it must hold that

$$x \geq \mathbf{E} \left[H(s, T, p) L(s, T, p) + \int_s^T H(s, u, p) (c(s, u, x, p) - \varepsilon(s, u, p)) du \right]$$

for any $x \geq L(s, s, p)$, where the latter follows since the process defined by equation (6) is a super-martingale. It is often the case that $L(s, T, p) = 0$ for all s and p . In these latter case the condition for the previous equation becomes

$$x \geq \mathbf{E} \left[\int_s^T H(s, u, p) (c(s, u, x, p) - \varepsilon(s, u, p)) du \right].$$

Next we explain the problem that we are interested to solve in this paper. We assume a minimal wealth structure L and an endowment rate evolution structure ε . The control stochastic problem that we propose to solve concerns a small investor that at time 0 has an initial capital x , is constrained to not let his wealth to fall below a minimal wealth process $L(0, \cdot, p)$, has a rate of endowment process, $\varepsilon(0, \cdot, p)$ and has at his disposal portfolio/consumption processes $(\pi, c) \in \mathcal{A}(L, \varepsilon)$. The following Proposition 1 is a direct consequence of Londoño [34, Theorem 2]; it provides conditions under which $\mathcal{A}(L, \varepsilon) \neq \emptyset$.

Proposition 1. Assume a minimal wealth structure L , and a rate evolution structure ε as in Definition 4. Assume that

$$H(s, t, p) L(s, t, p) - \int_s^t H(s, t, p) \varepsilon(s, t, p) du$$

is a martingale for all s, p . Then, there exist a cumulative consumption and endowment evolution structure $(X, 0, \varepsilon)$ with $(\pi, 0) \in \mathcal{A}(L, \varepsilon)$ where π is the portfolio on stocks defined by X .

Proof Define X by

$$X(s, t, x, p) \triangleq L(s, t, p) + (x - L(s, s, p))H^{-1}(s, t, p)$$

It follows using Londoño [34, Theorem 2] that $(X, 0, \varepsilon)$ is the desired cumulative consumption and endowment evolution structure. \square

The following condition is needed in order to solve the problem of optimal investment and consumption under less strict conditions on the minimal wealth structure. For the following condition let $(X, c, \varepsilon, \tau)$ be a rate of consumption and endowment evolution structure with *discounted payoff process* defined as

$$Y(s, t, x, p) \triangleq H(s, t, p)X(s, t, x, p) + \int_s^t H(s, u, p) (c(s, u, x, p) - \varepsilon(s, u, p)) du. \quad (6)$$

Condition 2. Let $(X, c, \varepsilon, \tau)$ be a rate of consumption and endowment evolution structure, as above. We assume that for all stopping times $\tau \in \mathcal{S}(X)$, and $0 \leq s \leq T$ the function

$$\varphi_{s, \tau}(x, p) = \mathbf{E}[Y(s, t \wedge \tau(s, x, p), x, p)]$$

is a continuous function in (x, p) , and the given family of functions is an equicontinuous set of functions on compact sets (in (x, p)), where $s \wedge t = \min(s, t)$ and it is assumed that $\sup \emptyset = \infty$. Here $\mathcal{S}(X)$ denotes, the family of stopping times that are (X, P) -consistent. Moreover assume that there exist positive constants $\gamma \geq 1$, $\alpha_1, \alpha_2, \alpha_3, \beta_0, \dots, \beta_n$, with $\alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1} + \sum_{i=0}^n \beta_i < 1$ such that the random field $Y(s, t, x, p)$ satisfies

$$\mathbf{E}[|Y(x, p, s, t) - Y(x', p', s', t')|^\gamma] \leq C \left(|s - s'|^{\alpha_1} + |t - t'|^{\alpha_2} + |x - x'|^{\alpha_3} + \sum_{i=0}^n |p_i - p'_i|^{\beta_i} \right).$$

This condition is usually satisfied when X is a process that solves a stochastic differential equation. For instance, see Kunita [29, Lemma 4.5.6]. The last inequality is needed in order to obtain a continuous modification of the random field and its conditional expectation. See Kolmogorov's continuity criterion for random fields (Kunita [29, Theorem 1.4.1 and Exercise 1.4.12]). In the following, conditional expectations of stochastic processes are the continuous modifications of the given stochastic processes.

For the problems of optimal consumption and terminal wealth that we describe below we shall assume that the minimal wealth structure is defined in a way such that the discounted minimal wealth process of an agent can not fall below the current value of future endowments,

$$L(s, t, p) = \frac{-1}{H(s, t, p)} \mathbf{E} \left[\int_t^T H(s, u, p) \varepsilon(s, u, p) du \mid \mathcal{F}_{s, t} \right], \quad (7)$$

for $0 \leq s \leq t \leq T$, $p \in \mathbb{R}_+^n$. In fact is not difficult to see that the family of stochastic processes defined by the last equation, is a minimal wealth process with $\mathcal{A}(L, \varepsilon) \neq \emptyset$, since the discounted payoff process $Y(s, t, p)$ satisfies

$$\begin{aligned} Y(s, t, p) &= H(s, t, p)L(s, t, p) - \int_s^t H(s, u, p)\varepsilon(s, u, p) \\ &= -\mathbf{E} \left[\int_s^T H(s, u, p)\varepsilon(s, u, p) du \mid \mathcal{F}_{s,t} \right] \end{aligned}$$

and therefore is clearly a martingale. In fact, Proposition 1 is a consequence of a more general theorem stated below. It allows to solve the problem of optimal consumption and investment under more general minimal wealth structures.

Theorem 1. *Let $(X, c, \varepsilon, \tau)$ be a rate of consumption and endowment evolution structure as in Definition 3 with cumulative endowment and consumption structures C , and E as defined by equations (3) and (4) respectively. If the family of processes defined by equation (6) are martingales for each x, p, s then $(X, E - C)$ is a hedgeable wealth-income structure. Moreover, if $Y(s, \cdot, x, p)$ is a super-martingales for each x, p and s , and Condition 2 holds, then $(X, E - C)$ is dominated by a hedgeable wealth-income structure $(X', E - C)$. See Londoño [34] for the definition of this concept. In particular, if a minimal wealth evolution structure L is given such that*

$$H(s, t, p)L(s, t, p) - \int_s^t H(s, u, p)\varepsilon(s, u, p)du$$

is a super-martingale for all s , and p with the property that for all $x \in \mathbb{R}$

$$x \geq L(s, s, p) + \mathbf{E} \left[\int_s^T H(s, u, p)c(s, u, x, p) du \right], \quad (8)$$

then there exist a hedgeable wealth and income evolution structure $(X', E - C)$ with portfolio on stocks π' , such that $X'(s, t, x, p) \geq L(s, t, p)$ for all s, p, t , and x , satisfying equation (8). In particular $(\pi', c) \in \mathcal{A}(L, \varepsilon)$.

Proof The first part of the proof is a straightforward consequence of Londoño [34, Theorem 2 and Theorem 3]. For the second part of the proof we observe that if X is defined by

$$\begin{aligned} X(s, t, x, p) &= L(s, t, p) + \frac{1}{H(s, t, p)} \mathbf{E} \left[\int_t^T H(s, u, p)c(s, u, x, p) du \mid \mathcal{F}_{s,t} \right] + \\ &\quad \frac{1}{H(s, t, p)} \left(x - L(s, s, p) - \mathbf{E} \left[\int_s^T H(s, u, p)c(s, u, x, p) du \right] \right), \end{aligned}$$

then $(X, E - C)$ is a wealth income evolution structure with the property that the process defined by the equation (6) above, is a super-martingale for any x ,

p , and s . As a consequence of the above there exist a hedgeable wealth and income evolution structure $(X', E - C)$ dominating $(X, E - C)$. It follows that for any x satisfying equation (8), $X'(s, t, x, p) \geq L(s, t, p)$ for all p, s and t . \square

It is clear from the previous theorem that it is still possible to consider minimal wealth structures more general than the ones we pursue in this paper (those that satisfy equation (7)).

4 Consumption and Portfolio Optimization

In this paper we are interested in solving the optimization problems presented in this section. We assume a state preference structure (U_1, U_2) . We also assume a minimal wealth-income structure L , defined as in equation (7), with an endowment rate evolution structure ε .

Problem 1 (Utility from consumption). *Under the hypotheses assumed here the problem of maximization of utility from consumption is defined to be the problem of maximizing of expected utility of discounted consumption,*

$$V_1(x, p) \triangleq \sup_{(\pi, c) \in \mathcal{A}_1(L, \varepsilon, x)} \mathbf{E} \int_0^T U_1(t, H(0, t, p)c(0, t, x, p)) dt,$$

for all p and $x > -\mathbf{E} \left[\int_0^T H(0, u, p)\varepsilon(0, u, p) du \right]$, where

$$\mathcal{A}_1(L, \varepsilon, x) \triangleq \left\{ (\pi, c) \in \mathcal{A}(L, \varepsilon) : \mathbf{E} \int_0^T U_1^-(t, H(0, t, p)c(0, t, x, p)) dt < \infty \right\}$$

and $U_1^-(t, x) = -(U_1(t, x) \wedge 0)$. We shall say that V_1 is the value function for the problem of optimization of utility from consumption.

Problem 2 (Utility from terminal wealth). *Under the hypotheses assumed in this section the problem of maximization of utility from terminal wealth is defined to be the problem of maximizing the expected utility from discounted terminal wealth at time T ,*

$$V_2(x, p) \triangleq \sup_{(\pi, c) \in \mathcal{A}_2(L, \varepsilon, x)} \mathbf{E} [U_2(H(0, T, p)X(0, T, x, p))],$$

for all p and $x > -\mathbf{E} \left[\int_0^T H(0, u, p)\varepsilon(0, u, p) du \right]$, where

$$\mathcal{A}_2(L, \varepsilon, x) \triangleq \{ (\pi, c) \in \mathcal{A}(L, \varepsilon) : \mathbf{E} [U_2^-(H(0, T, p)X(0, T, x, p))] < \infty \}$$

and $U_2^-(x) = -(U_2(x) \wedge 0)$. We shall say that V_2 is the value function for the problem of optimization of utility from terminal wealth.

Problem 3 (Utility from both consumption and terminal wealth). *Under the hypotheses assumed above the problem of maximization of utility from*

both consumption and terminal wealth is defined to be the problem of maximization of expected utility from consumption and terminal wealth,

$$V(x, p) \triangleq \sup_{(\pi, c, x) \in \mathcal{A}(L, \varepsilon, x)} \mathbf{E} \left[\int_0^T U_1(t, H(0, t, p) c(0, t, x, p)) dt + U_2(H(0, T, p) X(0, T, x, p)) \right]$$

for all p and $x > -\mathbf{E} \left[\int_0^T H(0, u, p) \varepsilon(0, u, p) du \right]$, where

$$\mathcal{A}(L, \varepsilon, x) \triangleq \mathcal{A}_1(L, \varepsilon, x) \cap \mathcal{A}_2(L, \varepsilon, x).$$

We shall say that V is the value function for the problem of optimization of utility from consumption and terminal wealth.

A few words are needed here. With the help of Proposition 1 it is also possible to consider more general optimization problems when the restriction on minimal wealth is not necessarily the current value of future endowments as defined by equation (7). However in this paper we do not pursue this line of research, and we believe this is an open area of research where more general restrictions on minimal wealth could be studied.

We also point out that 0 does not play any special role, and the concepts like wealth, cumulative income, portfolio process, state preference structure, value functions and alike can be carried out for any time interval $[s, T]$ with $0 \leq s \leq T$. The above remark allows us to consider parameterized utility preference structures with parameter $0 \leq s \leq T$, defined on the time interval $[s, T]$. This models how an agent can change preferences as time evolves. In Londoño [33] we study the investment and consumption behavior of agents that change preferences as the result of aging.

The problems considered above are different from the standard problems of optimal consumption and investment, see for instance (Karatzas and Shreve [24]). First, the optimization problems are over portfolio and consumptions which are *consistent*. Second, it looks at utility functions as reflecting the level of satisfaction over levels of consumption in Problem 1, final wealths in Problem 2, and on both in Problem 3, as valued by the market when the agent is making his consumption and investment decisions (at time 0).

Let us define

$$\Pi(s, t, p) \triangleq -\mathbf{E} \left[\int_t^T H(s, u, p) \varepsilon(s, u, p) du \mid \mathcal{F}_{s,t} \right] \quad (9)$$

For any $x > \Pi(t, t, p)$ we define $\mathcal{Y}(t, x, p)$ as the unique solution of

$$\mathcal{X}(t, \mathcal{Y}(t, x, p)) = x - \Pi(t, t, p)$$

where \mathcal{X} is defined by equation (2). It follows that $\mathcal{Y}(t, x, p) = \mathcal{X}^{-1}(t, x - \Pi(t, t, p))$.

Theorem 2. Assume the hypotheses of Problem 3, and in addition assume that (U_1, U_2) is a homogeneous state preference structure (see Condition 1). Define ξ as

$$\xi(s, t, x, p) \triangleq \begin{cases} H^{-1}(s, t, p) (\Pi(s, t, p) + \mathcal{X}(t, \mathcal{Y}(s, x, p))) & \text{if } x > \Pi(s, s, p) \\ H^{-1}(s, t, p) (\Pi(s, t, p) + x - \Pi(s, s, p)) & \text{otherwise,} \end{cases}$$

and let c be defined as

$$c(s, t, x, p) \triangleq \begin{cases} H^{-1}(s, t, p) I_1(t, \mathcal{Y}(s, x, p)) & \text{if } x > \Pi(s, s, p) \\ 0 & \text{otherwise.} \end{cases}$$

Then, (ξ, c, ε) is a hedgeable cumulative consumption and endowment structure, with portfolio $(\pi, c) \in \mathcal{A}(L, \varepsilon)$ that is optimal for the problem of optimal consumption and investment. The value function is given by

$$V(x, p) = G(0, \mathcal{Y}(0, x, p)),$$

where

$$G(s, y) = \int_s^T U_1(t, I_1(t, y)) dt + U_2(I_2(y))$$

for $0 < y < \infty$. The corresponding optimal portfolio on stocks is

$$(\xi(s, t, x, p) + \Pi(t, P(s, t, p)) + \phi_0(t, P(s, t, p))) (\sigma \sigma')^{-1} (b + \delta - r \mathbf{1}_n)(s, t, p) - (\phi_1(t, P(s, t, p)), \dots, \phi_n(t, P(s, t, p)))' \quad (10)$$

where

$$\Pi(t, p) \triangleq \Pi(t, t, p), \quad \phi_i(t, p) \triangleq p_i \frac{\partial \Pi(t, p)}{\partial p_i} \quad 0 \leq i \leq n$$

Proof Let us point out that Condition 1 implies that ξ is a (consistent) process, and clearly it is Lipschitz continuous. The homogeneity also implies that c is a (ξ, P) consistent process of class $C^{0,1}$. We observe that

$$\begin{aligned} Y(s, t, x, p) &\triangleq H(s, t, p) \xi(s, t, x, p) + \int_s^t H(s, u, p) (c(s, u, x, p) - \varepsilon(s, u, p)) du \\ &= x + \mathbf{E} \int_s^T H(s, u, p) \varepsilon(s, u, p) du - \mathbf{E} \left[\int_s^T H(s, u, p) \varepsilon(s, u, p) du \mid \mathcal{F}_{s,t} \right] \end{aligned} \quad (11)$$

is a martingale, and therefore Theorem 1 implies that (ξ, c, ε) is a cumulative consumption and endowment structure with portfolio $(\pi, c) \in \mathcal{A}(L, \varepsilon)$. Next, we observe that for $x > \Pi(s, s, p)$

$$\begin{aligned} \mathbf{E} \left[\int_s^T U_1(t, H(s, t, p) c(s, t, x, p)) dt \right] \\ + \mathbf{E} [U_2(H(s, T, p) \xi(s, T, x, p))] = G(s, \mathcal{Y}(s, x, p)) \end{aligned}$$

and, if (X', ϵ, c') is a hedgeable rate of consumption, endowment and wealth evolution structure, then for $x > \Pi(s, s, p)$,

$$\begin{aligned} \mathbf{E} \left[\int_s^T U_1(t, H(s, t, p) c'(s, t, x, p)) dt + U_2(H(s, T, p) X'(s, T, x, p)) \right] \leq \\ G(s, \mathcal{Y}(s, x, p)) - \mathcal{Y}(s, x, p) \left[\int_s^T I_1(t, \mathcal{Y}(s, x, p)) dt + I_2(\mathcal{Y}(s, x, p)) \right] \\ + \mathcal{Y}(s, x, p) \mathbf{E} \left[H(s, T, p) X'(s, T, x, p) + \int_s^T H(s, u, p) (c'(s, u, x, p)) du \right] \\ \leq G(s, \mathcal{Y}(s, x, p)) \end{aligned}$$

where the first inequality is a consequence of equation (1) and the last inequality is a consequence of the fact that the process defined by equation (6) is a super-martingale for any hedgeable wealth-income structure.

Next we prove that the optimal portfolio satisfies equation (10). It is known that the corresponding optimal portfolio should satisfy

$$\sigma'(s, t, p) \pi(s, t, x, p) = H^{-1}(s, t, p) \varphi(s, t, x, p) + \xi(s, t, x, p) \theta(s, t, p)$$

where $\varphi(s, t, x, p)$ is the process such that

$$Y(s, t, x, p) = x + \int_s^t \varphi'(s, u, x, p) dW_s(u)$$

Using the uniqueness of the decomposition of a continuous semimartingale as local martingale and a process of bounded variation, Ito's rule, and the fact that ε is a P consistent process, it follows by a straightforward computation that the optimal portfolio is given by equation (10). \square

Remark 1. If ε is a P consistent process where $\mathbf{E} \left[\int_s^T H(s, u, p) \varepsilon(s, u, p) du \mid \mathcal{F}_{s,t} \right]$ is a deterministic function then the proof of the above theorem shows that the optimal portfolio is

$$\pi(s, t, x, p) = (\sigma \sigma')^{-1} (b + \delta - r \mathbf{1}_n)(s, t, p) \xi(s, t, x, p)$$

One important example of the above case is when there are not additional income to invest in the portfolio.

Remark 2. One of the consequences of Theorem 2, is that the solution to Problem 3 above, under the hypothesis that the state preference structure is homogeneous, is also homogeneous in the sense that we explain next. For any time $0 \leq s \leq T$ the solution (ξ, c) of Problem 3 (as well as its associated optimal portfolio) satisfies the property that its restriction to the time interval $[s, T]$ is also optimal for the problem of optimal consumption and investment after time s (where the definition of the solution to the problem has been outlined after the definition of the solution to the problems of optimal consumption and investment). The latter remark is a consequence of the proof of Theorem 2.

Next, we present without proof the solution to the problem of optimal consumption and investment when there is not any preference on partial consumption of final wealth. The proof is similar to the proof of Theorem 2, and is left to the reader. In order to state the following theorem we introduce the functions

$$\mathcal{X}_1(t, y) = \int_t^T I_1(t', x) dt', \quad \mathcal{X}_2(y) = I_2(y)$$

and

$$G_1(t, x) = \int_s^T U_1(t, I_1(t', y)) dt' \quad G_2(y) = U_2(I_2(y))$$

for $y > 0$ and $0 \leq t \leq T$. We also set $\mathcal{Y}_1(t, x, p) = \mathcal{X}_1^{-1}(t, x - \Pi(t, t, p))$, and $\mathcal{Y}_2(t, x, p) = \mathcal{X}_2^{-1}(x - \Pi(t, t, p))$ for $x > 0$ and $0 \leq t \leq T$.

Theorem 3. Assume a homogeneous state preference structure and minimal wealth-income structure (U_1, U_2) , L defined as in equation (7), with an endowment rate evolution structure ε . Define ξ_1 as

$$\xi_1(s, t, x, p) \triangleq \begin{cases} H^{-1}(s, t, p) (\Pi(s, t, p) + \mathcal{X}_1(t, \mathcal{Y}_1(s, x, p))) & \text{if } x > \Pi(s, s, p) \\ H^{-1}(s, t, p) (\Pi(s, t, p) + x - \Pi(s, s, p)) & \text{otherwise,} \end{cases}$$

and let c_1 be defined as

$$c_1(s, t, x, p) \triangleq \begin{cases} H^{-1}(s, t, p) I_1(t, \mathcal{Y}_1(s, x, p)) & \text{if } x > \Pi(s, s, p) \\ 0 & \text{otherwise.} \end{cases}$$

Then, $(\xi_1, c_1, \varepsilon)$ is a cumulative consumption and endowment structure, with portfolio $(\pi_1, c_1) \in \mathcal{A}(L, \varepsilon)$ that is optimal for the problem of optimal consumption and investment. The value function is given by

$$V_1(x, p) = G_1(0, \mathcal{Y}_1(0, x, p)),$$

for $0 < y < \infty$, and optimal portfolio on stocks

$$(\xi_1(s, t, x, p) + \Pi(t, P(s, t, p)) + \phi_0(t, P(s, t, p))) (\sigma \sigma')^{-1} (b + \delta - r \mathbf{1}_n)(s, t, p) - (\phi_1(t, P(s, t, p)), \dots, \phi_n(t, P(s, t, p)))',$$

where Π , and ϕ_i , $0 \leq i \leq n$ are defined as in Theorem 2.

Theorem 4. Assume a homogeneous state preference structure and minimal wealth-income structure, L defined as in equation (7), with an endowment rate evolution structure ε . Let ξ_2 be defined as

$$\xi_2(s, t, x, p) \triangleq \begin{cases} H^{-1}(s, t, p) (\Pi(s, t, p) + \mathcal{X}_2(t, \mathcal{Y}_2(s, x, p))) & \text{if } x > \Pi(s, s, p) \\ H^{-1}(s, t, p) (\Pi(s, t, p) + x - \Pi(s, s, p)) & \text{otherwise.} \end{cases}$$

Then, $(\xi_2, 0, \varepsilon)$ is a cumulative consumption and endowment structure, with portfolio $(\pi_2, 0) \in \mathcal{A}(L, \varepsilon)$ that is optimal for the problem of optimal consumption and investment. The value function is given by

$$V(x, p) = G(\mathcal{Y}_2(0, x, p)),$$

for $0 < y < \infty$, and the optimal portfolio on stocks is

$$(\xi_2(s, t, x, p) + \Pi(t, P(s, t, p)) + \phi_0(t, P(s, t, p))) (\sigma \sigma')^{-1} (b + \delta - r \mathbf{1}_n)(s, t, p) - (\phi_1(t, P(s, t, p), \dots, \phi_n(t, P(s, t, p)))' \quad (12)$$

where Π , and ϕ_i , $0 \leq i \leq n$ are defined as in Theorem 2.

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